AI: Vector Basics

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So what on earth is a vector?

Vectors are the bread and butter for AI/ML as well as computer graphics. Getting a fundamental understanding is a critical building block in understanding other concepts.

In 2D space, a vector `v` is nothing but a point `(x1, x2)` on the 2D plot. The vector is drawn as a line that starts from origin and extends till `(x1, x2)`.

The magnitude of that vector is obtained using Pythagoras formula: `|v| = √(x1² + x2²)`

A vector in 2D space can be split into its sine and cosine components using trigonometry. Given a vector with an angle of theta relative to the x-axis, the vector is made of 2 components:

1. The cosine component is equal to the magnitude of the vector multiplied by the cosine of theta.

2. The sine component is equal to the magnitude of the vector multiplied by the sine of theta.

Visually, this
looks like:
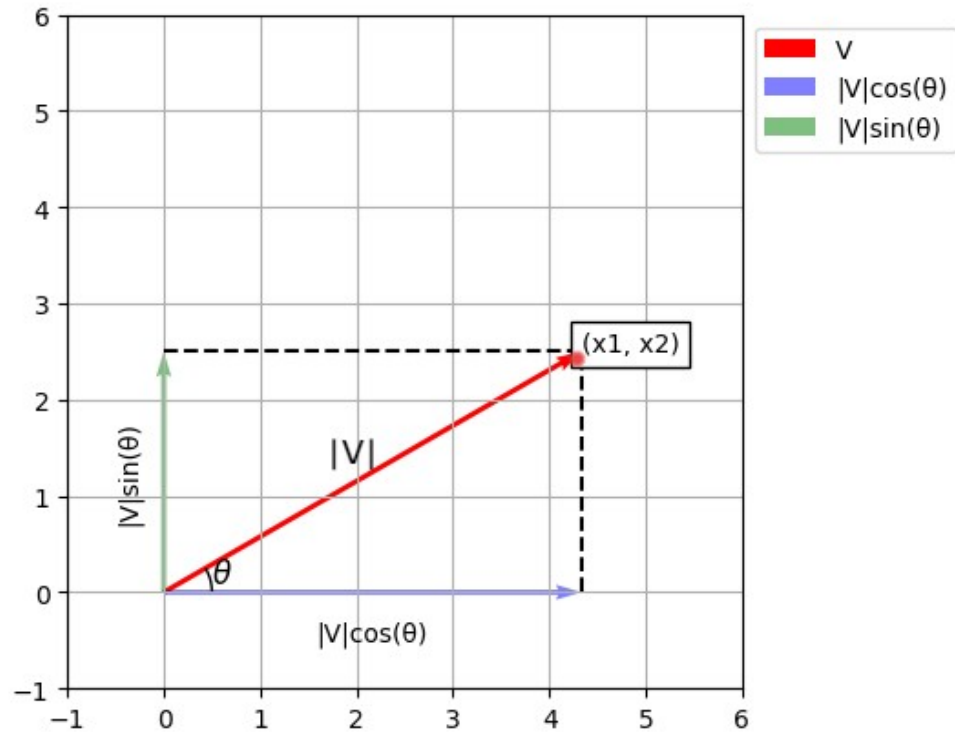
Example of a vector with its magnitude, angle, sine and cos components.

A vector can also be considered as an object which has both a magnitude and a direction. The coordinate of the vector `(x1, x2)` determines both its magnitude and direction.

A vector between 2 different points

Given 2 vectors, it is also possible to obtain a 3rd vector from as below. This is illustrated below:
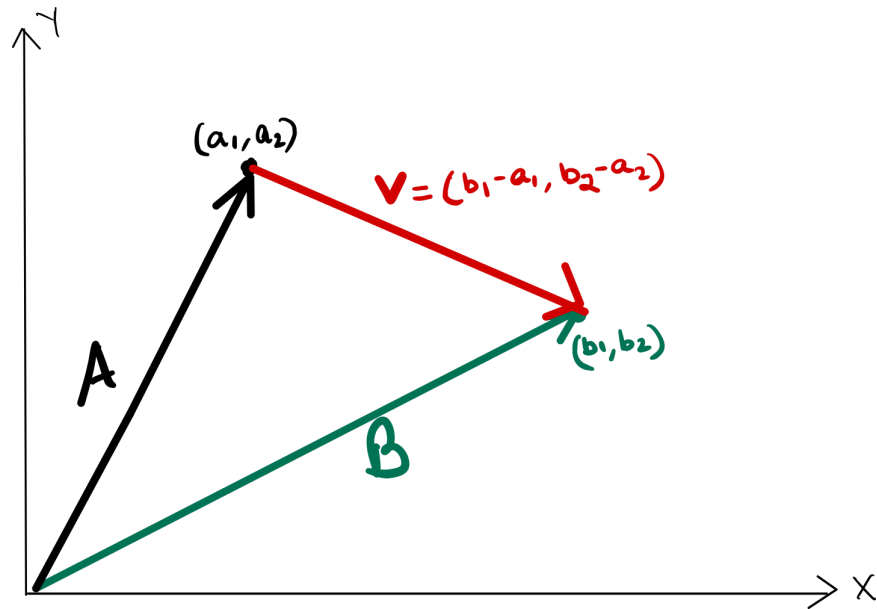
Here, vector `v` is obtained from vectors `A` and `B` by taking the differences between their coordinates. The coordinates of vectors `A` and `B` both determine `v`'s direction.

> 💡 An application of the concept of generating a new vector from 2 vectors is used in SVM machines. To determine the distance across hyperplanes, a new vector is drawn between 2 points, each point lying on one of the support vectors.

Okay, and what is a vector dot product?

The dot product of 2 vectors is nothing but the product of the magnitude of the 2 vectors.

However, it is only the product of the magnitudes of the components of both vectors that are pointing in the same direction.

How do we find the components of the 2 vectors that point in the same direction? There's a trick.

Say we want to find the dot product of 2 vectors, `A` and `B`.

If we consider `B` to be the x-axis of vector `A`'s plot, then as we saw earlier, `A * cos(θ)` is the component in the direction of `B`.

Thus the dot product of vector `A` and `B` is: `|A| * cos(θ) * |B|` ; or in other words:

`A.B = |A| * |B| * cos(θ)` .

It is easy to see that, if θ is 90 degrees, the dot product is 0. That's because there's no component of A that's in the direction of B . Further, θ value of 0 degrees makes the dot product simply the product of the magnitudes of A and B .

This is also known as the "Trigonometric definition" of the dot product.

The dot product and projections

The dot product is also useful for calculating projections. Let's say we have a vector A and we want to project it onto vector B . The projection of A onto B is given by the formula: `proj_B(A) = (A.B / |B|²) * B`

This formula is essentially taking the dot product of A and B , dividing by the magnitude of B squared, and then multiplying by B . The resulting vector is the projection of A onto B . This is useful in various machine learning and computer graphics applications, such as projecting an image onto a surface or projecting a high-dimensional data point onto a lower-dimensional space.

Note that the projection is itself a vector in the same direction as B , although it is a scaled version of the vector B .

Let's try to intuitively understand how to obtain that projection equation.

1. The stuff on the RHS in brackets is called the scaling factor. That's what we scale the vector B by.

2. Using the dot product formula, The scaling factor can also be written as: `|A| cos θ / |B|`

3. This is basically the ratio of the length (magnitude) of the component of A that lies along B (which is `|A| cos θ`) to the length of B (which is `|B|`). This gives us the fraction of B 's length that A 's projection will take up.

4. Finally, we multiply this by B to get the actual projection of A onto B , as a vector.

So why don't we just use the simplified formula in step 2 instead of the dot product in the earlier formula?

That's because the dot product has another form that makes it easy to calculate. If we know the features (coordinates) of 2 vectors, we don't have to waste resources

calculating the angle between them, just to find the projection. We can just use the formula:

`A.B = ∑i(Ai * Bi)` (This is also known as the "Algebraic definition" of the dot product).

And then plug this into the dot product version of the `proj_B(A)` equation.

This is the form of the dot product that is often used in machine learning and other applications. It is simply the sum of the product of the corresponding coordinates of the two vectors. This makes it easier to calculate, especially in high-dimensional spaces where calculating angles and magnitudes can be computationally expensive.

Once we obtain the dot product, we can use the formula for calculating the projection of `A` onto `B`. The dot product gives us the numerator of the formula. We can thus easily calculate the projection without having to calculate the angle between the two vectors.

> 💡 Note that the magnitude of the projection is simply `|A|cos(θ)`, where `θ` is the angle between vectors `A` and `B`. In most cases, we don't know or care about the angle between `A` and `B` and we just have `A`'s and `B`'s coordinates. So we can simply rewrite the `|A|cos(θ)` equation using the dot product, as `A.B / |B|`. This dot product technique is used for the computing magnitude of projections in ray tracing and machine learning SVMs!

Proof that the Algebraic and Trigonometric definitions are equivalent

Let's use the `arctan2` function for this. The `arctan2` function is nothing but the angle that a vector makes with respect to X-axis. The `arctan2` of a vector `|V|` with coordinations `(x1, x2)` is simply written as `arctan2(x1, x2)`. You don't need to know how to calculate the `arctan2` for the proof, just know that it is the angle between the vector `|V|` and the X-axis.

Let's use this knowledge to translate the dot product from one form to another. Specifically, let's find the dot product between two vectors `|A|` and `|B|`.

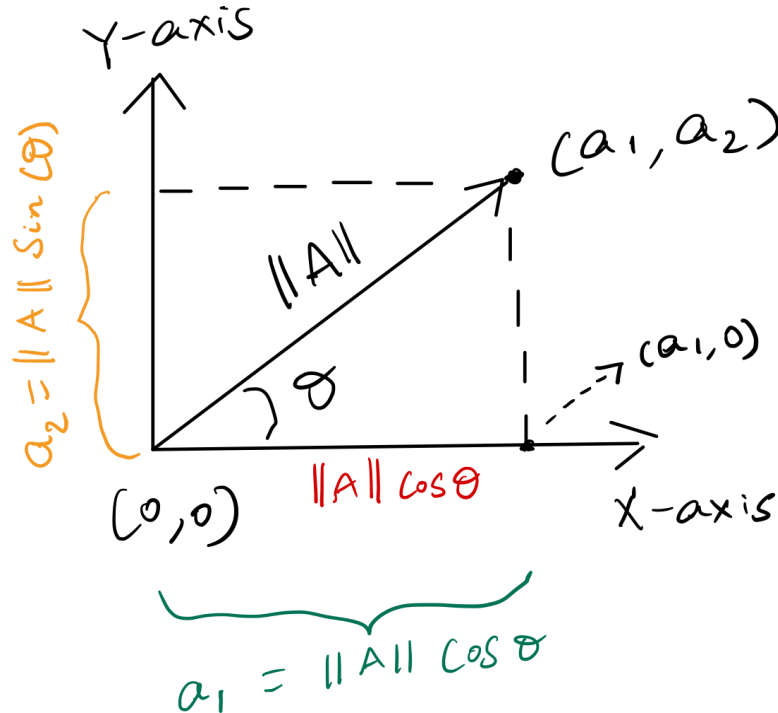The angle between two vectors `|A|` with coordinates `(a1, a2)` and `|B|` with `(b1, b2)` is:

`θ(AB) = arctan2(a1, a2) - arctan2(b1, b2) = θ(A) - θ(B)`

`Cos(θ(AB)) = Cos(θ(A) - θ(B))`

Using the Cosine formula: `Cos(P-Q) = Cos(P)CosQ + Sin(P)Sin(Q)`, we get:

`Cos(θ(AB)) = Cos(θ(A))Cos(θ(B)) + Sin(θ(A))Sin(θ(B))`

Let's fine `Cos(θ(A))`, it looks like this visually:

Relationship of Vector's [co]sine components with its coordinate points.

As you can see, `a1 = |A| Cos(θ(A))`

Thus, `Cos(θ(A)) = a1 / |A|`.

Using this in the equation for `Cos(θ(AB))`, we get:

`Cos(θ(AB)) = (a1 / |A|) * (b1 / |B|) + (a2 / |A|) * (b2 / |B|)`

That can be written as:

`|A| * |B| * Cos(θ(AB)) = (a1 * b1) + (a2 * b2)`

Hence, both the algebraic and trigonometric forms of the dot product are equivalent.

Conclusion

This article just covered a few basics of vectors. There's obviously much more about vectors such as cross products and so forth, we will revisit this article in the future to go over them as needed.